Name ____ MTH 1002 - Calculus 2

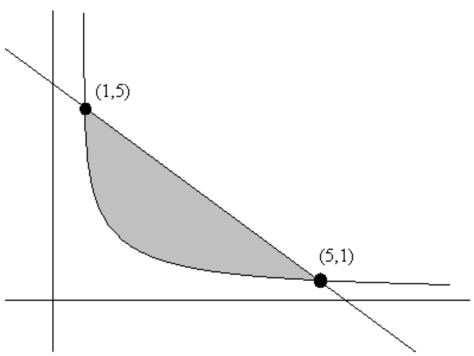
PRACTICE EDVANCED STANDING EXAM

No books, no notes and no calculators permitted. **Show all work.**

[200 pts total]

1. Make a sketch and label the intersection points of the region bounded by $x + y = 6$ and $y = \dfrac{5}{x}$.

   Set up the integral and evaluate to find the area of the region. [9 pts]



$$x + y = 6 \Rightarrow y = -x + 6$$
$$-x + 6 = \tfrac{5}{x} \Rightarrow -x^2 + 6x = 5$$
$$x^2 - 6x + 5 = 0 \Rightarrow (x-5)(x-1) = 0 \Rightarrow x = 5 \text{ and } x = 1$$
$$\int [T] - [B]\,dx$$
$$\int_{x=1}^{x=5} [-x+6] - [\tfrac{5}{x}]\,dx = \int_{x=1}^{x=5} -x + 6 - 5x^{-1}\,dx$$

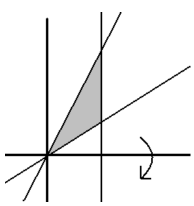
5 pts formula/setup

$$= -\tfrac{1}{2}x^2 + 6x - 5\ln x \Big]_{x=1}^{x=5}$$
$$= [-\tfrac{25}{2} + 30 - 5\ln 5] - [-\tfrac{1}{2} + 6 - 5\ln 1]$$
$$= -\tfrac{25}{2} + 30 - 5\ln 5 + \tfrac{1}{2} - 6 + 0$$
$$= 24 - \tfrac{24}{2} - 5\ln 5 = 24 - 12 - 5\ln 5 = 12 - 5\ln 5$$

4 pt evaluate

2. Setup the integral to find the volume of the solid generated when the area between the curves of $y = 2x$, $y = 5x$, $x = 0$, and $x = 1$ is rotated around the *x-axis*. [9 pts]

(NOTE: You do NOT have to evaluate the integral and get a number. Just setup the integral. That is enough.)



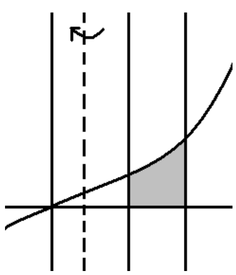
By Disks : $\int_{x=0}^{x=1} \pi\left([5x]^2 - [2x]^2\right)dx$ or $\int_{x=0}^{x=1} \pi\left(25x^2 - 4x^2\right)dx = \int_{x=0}^{x=1} 21\pi x^2\,dx$

or 9 pts

By Shells : $\int_{y=0}^{y=2} 2\pi y\left([\tfrac{1}{2}y] - [\tfrac{1}{5}y]\right)dy + \int_{y=2}^{y=5} 2\pi y\left([1] - [\tfrac{1}{5}y]\right)dy$

3. Setup the integral to find the volume of the solid generated when the region bounded by the graphs of $y = x^3$, the *x-axis*, $x = 4$, and $x = 5$ is rotated around the line $x = 2$. [9 pts]

(NOTE: You do NOT have to evaluate the integral and get a number. Just setup the integral. That is enough.)



Off - Center Shells : $\int_{x=4}^{x=5} 2\pi(x-2)x^2\,dx$ 9 pts

Or

Off - Center Disks : $\int_{y=64}^{y=125} \pi\left([5-1]^2 - [\sqrt[3]{y} - 1]^2\right)dy + \int_{y=0}^{y=64} \pi\left([5-1]^2 - [4-1]^2\right)dy$

Evaluate the following antiderivatives:

4. $\int xe^{2x}\,dx$ $\qquad u = x \quad du = 1dx \qquad v = \frac{1}{2}e^{2x} \qquad dv = e^{2x}dx$

[10 pts]

$$\int xe^{2x}dx = x\cdot\tfrac{1}{2}e^{2x} - \int \tfrac{1}{2}e^{2x}\cdot 1dx$$ 6 pts formula/setup

$$\tfrac{1}{2}xe^{2x} - \int \tfrac{1}{2}e^{2x}dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C$$ 4 pt final antideriv & ans

5. $\int \cos^8 x \sin^3 x\,dx$

[10 pts]

$$\int \cos^8 x\cdot\sin^2 x\cdot\sin x dx = \int \cos^8 x\cdot\left(1-\cos^2 x\right)\cdot\sin x dx$$ 3 pts split off one sin & convert others

$$u = \cos x \qquad \tfrac{du}{dx} = -\sin x \Rightarrow -1du = \sin x dx \Rightarrow\Rightarrow \int u^8\cdot\left(1-u^2\right)\cdot -1du$$ 3 pts u-sub (omit if direct to antideriv)

$$\int -u^8\cdot\left(1-u^2\right)du = \int -u^8 + u^{10}du = -\tfrac{1}{9}u^9 + \tfrac{1}{11}u^{11} = -\tfrac{1}{9}(\cos x)^9 + \tfrac{1}{11}(\cos x)^{11} + C$$ 2 pts antideriv & 2 pts back to x's

6. $\int \dfrac{1}{x^2\sqrt{81+x^2}}dx$

[15 points]

If $\quad x = 9\tan\theta \qquad$ then $\quad dx = 9\sec^2\theta d\theta$

3 pts trig choice

$$\int \frac{1}{(9\tan\theta)^2\sqrt{81+(9\tan\theta)^2}}9\sec^2\theta\,d\theta$$ 3 pts good sub

$$\int \frac{1}{81\tan^2\theta\sqrt{81+81\tan^2\theta}}9\sec^2\theta\,d\theta = \int \frac{1}{81\tan^2\theta\sqrt{81(1+\tan^2\theta)}}9\sec^2\theta\,d\theta = \int \frac{1}{81\tan^2\theta\sqrt{81\sec^2\theta}}9\sec^2\theta\,d\theta = \int \frac{1}{81\tan^2\theta|9\sec\theta|}9\sec^2\theta\,d\theta$$ (OK to omit abs val)

$$= \int \frac{1}{81\tan^2\theta\cdot 9\sec\theta}9\sec^2\theta\,d\theta = \int \frac{\sec\theta}{81\tan^2\theta}\,d\theta = \int \frac{1}{81}\cdot\frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta = \int \frac{1}{81}\cdot\frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{81}\cdot\frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}d\theta$$ 3 pts reduce correctly after sub

$$= \int \tfrac{1}{81}\csc\theta\cot\theta d\theta = -\tfrac{1}{81}\csc\theta + C$$ 3 pts antideriv

$$x = 9\tan\theta \;\Rightarrow\; \tan\theta = \tfrac{x}{9} \;\Rightarrow\; \text{OPP} = x \text{ and } \text{ADJ} = 9 \text{ and } \text{HYP} = \sqrt{81+x^2}$$ (OK to explain by drawing triangle)

$$= -\tfrac{1}{81}\csc\theta + C$$

$$= -\tfrac{1}{81}\cdot\frac{\sqrt{81+x^2}}{x} + C$$ 3 pts back to x's

7. $\int \frac{4x^2+2x+3}{x(x^2+1)}dx$

[15 pts]

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}=\frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

$$4x^2+2x+3=A(x^2+1)+(Bx+C)x$$

$$@\,x=0 \;\Rightarrow\; 3=A(1)+(0+C)\cdot 0 \;\Rightarrow\; 3=A+0 \;\Rightarrow\; A=3$$

$$@\,x=1 \;\Rightarrow\; 9=A(2)+(B+C)\cdot 1 \;\Rightarrow\; 9=2[3]+B+C \;\Rightarrow\; 9=6+B+C \;\Rightarrow\; B+C=3$$

$$@\,x=-1 \;\Rightarrow\; 5=A(2)+(-B+C)\cdot -1 \;\Rightarrow\; 5=2[3]+B-C \;\Rightarrow\; 5=6+B-C \;\Rightarrow\; B-C=-1$$

9 pts decomp

$$B+C=3$$
$$B-C=-1$$
$$2B=2 \;\Rightarrow\; B=1 \;\Rightarrow\; C=2$$

$$\int \frac{4x^2+2x+3}{x(x^2+1)}dx=\int \tfrac{3}{x}+\tfrac{1x+2}{x^2+1}\,dx$$

$$=\int \tfrac{3}{x}+\tfrac{x}{x^2+1}+\tfrac{2}{x^2+1}\,dx=\int 3\cdot\tfrac{1}{x}+\tfrac{1}{2}\cdot\tfrac{2x}{x^2+1}+2\cdot\tfrac{1}{x^2+1}\,dx$$

$$=3\ln(x)+\tfrac{1}{2}\ln(x^2+1)+2\,\mathrm{Arctan}(x)+C$$

6 pts integration

8. $\int_4^{+\infty}\frac{1}{\sqrt{x}}dx$

[9 pts]

$$\int_4^{+\infty}\tfrac{1}{\sqrt{x}}\,dx=\lim_{L\to+\infty}\int_4^{L}\tfrac{1}{\sqrt{x}}\,dx$$ 3 pts rewrite as limit

$$\lim_{L\to+\infty}\int_4^{L}x^{-\frac{1}{2}}dx=\lim_{L\to+\infty}2x^{\frac{1}{2}}\Big]_4^L=\lim_{L\to+\infty}2\sqrt{x}\Big]_4^L$$ 3 pts antidiff

$$=\lim_{L\to+\infty}\left(2\sqrt{L}-2\sqrt{4}\right)=\lim_{L\to+\infty}\left(2\sqrt{L}-4\right)\Rightarrow 2\sqrt{\infty}-4\Rightarrow\infty-4\Rightarrow\infty\Rightarrow \text{Diverge}$$ 3 pts take lim & final ans

Solve the following differential equation by the method of integrating factors.

9. $x\dfrac{dy}{dx}+3y=\dfrac{\sec^2 x}{x^2}$

[12 pts]

$\frac{dy}{dx}+\frac{3}{x}y=\frac{\sin x}{x^3}$

I.F. $\Rightarrow e^{\int P}=e^{\int \frac{3}{x}dt}=e^{3\ln x}=e^{\ln x^3}=x^3$

$\Rightarrow x^3\cdot\left(\frac{dy}{dx}+\frac{3}{x}y\right)=x^3\cdot\left(\frac{\sec^2 x}{x^3}\right)$

$x^3\frac{dy}{dx}+3x^2y=\sec^2 x$

$\left(x^3y\right)'=\sec^2 x$

$x^3y=\int\sec^2 x dx$

$x^3y=\tan x+C$

$y=\frac{\tan x}{x^3}+\frac{C}{x^3}$

2 pts divide & 5 pts int fact & 5 pts rewrite/integrate

10. Setup, but do ***not*** evaluate, an integral to find the arclength of the following:

$x=y^2+5$ from $(6,1)$ to $(14,3)$

[9 pts]

$x=y^2+5\Rightarrow\frac{dx}{dy}=2y$ or $x=y^2+5\Rightarrow y=(x-5)^{\frac{1}{2}}\Rightarrow\frac{dy}{dx}=\frac{1}{2}(x-5)^{-\frac{1}{2}}$

$\int_{y=1}^{y=3}\sqrt{1+[2y]^2}\,dy=\int_{y=1}^{y=3}\sqrt{1+4y^2}\,dy$

$\int_{x=6}^{x=14}\sqrt{1+\left[\frac{1}{2}(x-5)^{-\frac{1}{2}}\right]^2}\,dx=\int_{x=6}^{x=14}\sqrt{1+\frac{1}{4}(x-5)^{-1}}\,dx$

9 pts (2 pts limits & 3 pts root and "1+" term & 4 pts deriv squared)

11. Set-up and evaluate (you ***do*** have to complete the integral and get a numeric answer) the definite integral that represents the area of the surface generated by revolving the given curve about the *y*-axis:

$x=3t$ and $y=4t$ with $0\le t\le 2$

[10 pts]

$x=3t\Rightarrow\frac{dx}{dt}=3$

$y=4t\Rightarrow\frac{dy}{dt}=4$

$\int_{t=0}^{t=2}2\pi(3t)\sqrt{[3]^2+[4]^2}\,dt=\int_{t=0}^{t=2}6t\pi\sqrt{9+16}\,dt$

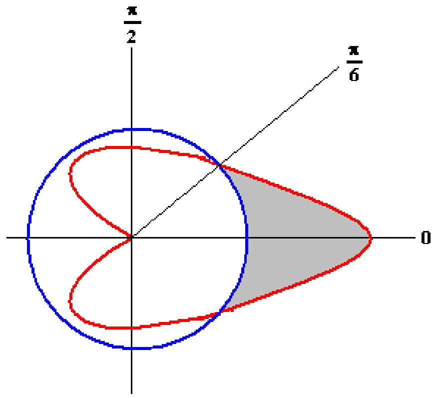
$=\int_{t=0}^{t=2}6t\pi\sqrt{25}\,dt=\int_{t=0}^{t=2}6t\pi\cdot 5\,dt=\int_{t=0}^{t=2}30t\pi\,dt$

$=15t^2\pi\Big]_{t=0}^{t=2}=[60\pi]-[0]=60\pi$

10 pts (2 pts derivs & 2 pts setup & 4 pts antidiff properly & 2 pts plugin)

12. Graph the polar functions $r = 3$ and $r = 2 + 2\cos\theta$ and then set-up and evaluate (you ***do*** have to complete the integral and get a numeric answer) an integral to find the area of the region that is outside the circle and inside the cardioid**.**

[16 pts]



Intersection: $2 + 2\cos\theta = 3$

$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ in QI & QVI $\Rightarrow \theta = \frac{\pi}{6}$ & $\theta = -\frac{\pi}{6}$

16 pts (6 pts graphs & 4 pts setup integral & 3 pts card area & 2 pts circle area & 1 pt final answer)

Total Area: $A = 2\cdot\left(A_{\text{card}} - A_{\text{circ}}\right) = 2\cdot\left(\int_{\theta=0}^{\theta=\frac{\pi}{6}} \frac{1}{2}\left(2+2\cos\theta\right)^2 d\theta - \int_{\theta=0}^{\theta=\frac{\pi}{6}} \frac{1}{2}(3)^2 d\theta\right)$

$A_{\text{card}} = \int_{\theta=0}^{\theta=\frac{\pi}{6}} \frac{1}{2}\left(2+2\cos\theta\right)^2 d\theta$

$A_{\text{card}} = \int_{\theta=0}^{\theta=\frac{\pi}{6}} \frac{1}{2}\left(4+8\cos\theta+4\cos^2\theta\right)d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{6}} \left(2+4\cos\theta+2\cos^2\theta\right)d\theta$

$= \int_{\theta=0}^{\theta=\frac{\pi}{6}} \left(2+4\cos\theta+2\cdot\frac{1}{2}\left[1+\cos(2\theta)\right]\right)d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{6}} \left(2+4\cos\theta+1+\cos(2\theta)\right)d\theta$

$= \int_{\theta=0}^{\theta=\frac{\pi}{6}} \left(3+4\cos\theta+\cos(2\theta)\right)d\theta = 3\theta + 4\sin\theta + \frac{1}{2}\sin(2\theta)\Big|_{\theta=0}^{\theta=\frac{\pi}{6}}$

$= \left(3\left[\frac{\pi}{6}\right] + 4\sin\left[\frac{\pi}{6}\right] + \frac{1}{2}\sin\left(2\left[\frac{\pi}{6}\right]\right)\right) - \left(3[0] + 4\sin[0] + \frac{1}{2}\sin(2[0])\right)$

$= \left(\frac{\pi}{2} + 4\sin\left(\frac{\pi}{6}\right) + \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right) - (0+0+0)$

$= \frac{\pi}{2} + 4\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\pi}{2} + 2 + \frac{\sqrt{3}}{4} = \frac{2\pi}{4} + \frac{8}{4} + \frac{\sqrt{3}}{4} = \frac{2\pi+8+\sqrt{3}}{4}$

$A_{\text{circ}} = \int_{\theta=0}^{\theta=\frac{\pi}{6}} \frac{1}{2}(3)^2 d\theta$

$A_{\text{circ}} = \int_{\theta=0}^{\theta=\frac{\pi}{6}} \frac{9}{2} d\theta = \frac{9}{2}\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{6}}$

$= \left(\frac{9}{2}\left[\frac{\pi}{6}\right]\right) - \left(\frac{9}{2}[0]\right)$

$= \frac{3\pi}{4} - 0 = \frac{3\pi}{4}$

Total Area: $A = 2\cdot\left(A_{\text{card}} - A_{\text{circ}}\right) = 2\cdot\left(\frac{2\pi+8+\sqrt{3}}{4} - \frac{3\pi}{4}\right) = 2\cdot\left(\frac{8-\pi+8+\sqrt{3}}{4}\right) = \frac{8-\pi+8+\sqrt{3}}{2}$

13. Find an approximation of the definite integral $\int_0^1 e^{-x^3}\,dx$ by utilizing a series expansion: [12 pts total]

(Hint: Simply write the first four nonzero terms that will be used in the evaluation of the answer. You do ***not*** have to actually convert the resulting fractions to their decimal form or combine them in any way.)

$$e^x = 1 + \tfrac{1}{1!}x^1 + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \ldots$$

3 pts for initial series

$$e^{-x^4} = 1 + \tfrac{1}{1!}\left(-x^3\right)^1 + \tfrac{1}{2!}\left(-x^3\right)^2 + \tfrac{1}{3!}\left(-x^3\right)^3 + + \ldots$$

$$= 1 - \tfrac{1}{1!}x^3 + \tfrac{1}{2!}x^6 - \tfrac{1}{3!}x^9 + \ldots$$

3 pts for making new series

$$\int e^{-x^3}\,dx = \int \left(1 - \tfrac{1}{1!}x^3 + \tfrac{1}{2!}x^6 - \tfrac{1}{3!}x^9 + \ldots\right)dx =$$

$$= x - \tfrac{1}{1!\cdot 4}x^4 + \tfrac{1}{2!\cdot 7}x^7 - \tfrac{1}{3!\cdot 10}x^{10} + \ldots$$

$$= x - \tfrac{1}{1\cdot 4}x^4 + \tfrac{1}{2\cdot 7}x^7 - \tfrac{1}{6\cdot 10}x^{10} + \ldots$$

$$= x - \tfrac{1}{4}x^4 + \tfrac{1}{14}x^7 - \tfrac{1}{60}x^{10} + \ldots$$

3 pts for integrating term-by-term

$$\int_0^1 e^{-x^3}\,dx = \left(x - \tfrac{1}{4}x^4 + \tfrac{1}{14}x^7 - \tfrac{1}{60}x^{10} + \ldots\right)\Big|_0^1$$

$$= \left([1] - \tfrac{1}{4}[1]^4 + \tfrac{1}{14}[1]^7 - \tfrac{1}{60}[1]^{10} + \ldots\right) - \left([0] - \tfrac{1}{4}[0]^4 + \tfrac{1}{14}[0]^7 - \tfrac{1}{60}[0]^{10} + \ldots\right)$$

$$= \left(1 - \tfrac{1}{4} + \tfrac{1}{14} - \tfrac{1}{60} + \ldots\right) - \left(0 - 0 + 0 - 0 + \ldots\right)$$

$$= 1 - \tfrac{1}{4} + \tfrac{1}{14} - \tfrac{1}{60} + \ldots$$

3 pts for plugging in and subtracting and final answer

14. Find the first four nonzero terms of the Taylor Series for $f(x)=\cos(x)$ about $x=\frac{\pi}{6}$.
[12 points]

$f^{(k)}(x)$	$f^{(k)}(c)$	$c_k = f^{(k)}(c)/k!$
$f(x)=\cos(x)$	$\cos(\frac{\pi}{6})=\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}/0! = \frac{\sqrt{3}}{2}/1 = \frac{\sqrt{3}}{2}$
$f'(x)=-\sin(x)$	$-\sin(\frac{\pi}{6})=-\frac{1}{2}$	$-\frac{1}{2}/1! = -\frac{1}{2}/1 = -\frac{1}{2}$
$f''(x)=-\cos(x)$	$-\cos(\frac{\pi}{6})=-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}/2! = -\frac{\sqrt{3}}{2}/2 = -\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = -\frac{\sqrt{3}}{4}$
$f'''(x)=+\sin(x)$	$\sin(\frac{\pi}{6})=\frac{1}{2}$	$\frac{1}{2}/3! = \frac{1}{2}/6 = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12}$

$$\cos(x)=\frac{\sqrt{3}}{2}-\frac{1}{2}\left(x-\frac{\pi}{6}\right)^1-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)^2+\frac{1}{12}\left(x-\frac{\pi}{6}\right)^3+\ldots$$

3 pts each term (2 pts per coeff & 1 pt *x*-terms)

Determine whether the following **geometric series** converges or diverges. If it converges, evaluate the sum.

15. $$\sum_{n=0}^{+\infty}\left[2^{3n}\cdot 3^{1-2n}\right]$$

[9 points]

$$\sum_{n=0}^{+\infty}2^{3n}\cdot 3^1\cdot 3^{-2n} = \sum_{n=0}^{+\infty}3\cdot\frac{(2^3)^n}{(3^2)^n} = \sum_{n=0}^{+\infty}3\cdot\frac{8^n}{9^n}$$

$$=\sum_{n=0}^{+\infty}3\cdot\left(\frac{8}{9}\right)^n \Rightarrow \left|\frac{8}{9}\right| = \frac{8}{9} < 1 \Rightarrow \text{converge}$$

$$\text{Sum}: \frac{[3]}{1-[8/9]} = \frac{3}{1/9} = \frac{3}{1}\cdot\frac{9}{1} = 27$$

2 pts find *a* and *r*-terms by rewriting or expanding series & 4 pts conv/div explanation & 3 pts sum

Determine whether the following series Converges Absolutely, Converges Conditionally, or Diverges. State the convergence test used (Limit Comparison Test, Ratio Test, etc.) or identify the important features of any series you use (example: "*p*-series with *p*=…") for each step to explain your reasoning.

16. $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+3}{n^3+1}$$

[9 points]

$|a_n| = \frac{n^2+3}{n^3+1} \Rightarrow |a_{n+1}| = \frac{(n+1)^2+3}{(n+1)^3+1} = \frac{n^2+\dots}{n^3+\dots}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^2+\dots}{n^3+\dots}\cdot\frac{n^3+1}{n^2+3} = \lim_{n\to\infty}\frac{n^5+\dots}{n^5+\dots} = \frac{1}{1} = 1 \Rightarrow$ no information

Split up into $\sum +$ and $\sum \pm$ use other tests.

$\sum \pm$: $\quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+3}{n^3+1}$

By Alternating Series Test

1. Check decreasing

By derivative method : $a_n = \frac{n^2+3}{n^3+1} \Rightarrow f(x) = \frac{x^2+3}{x^3+1}$

$f'(x) = \frac{2x(x^3+1)-3x^2(x^2+3)}{(x^3+1)^2} = \frac{2x^4+2x-3x^4-9x^2}{(x^3+1)^2}$

$= \frac{-2x^4-9x^2+2x}{(x^3+1)^2} = \frac{-}{+} = - \Rightarrow$ Decreasing

2. Check terms $\to 0$

$\lim_{n\to\infty}\frac{n^2+3}{n^3+1} \overset{L'Hop}{=} \lim_{n\to\infty}\frac{2n}{3n^2} = \lim_{n\to\infty}\frac{2}{3n} \Rightarrow \frac{2}{\infty} \Rightarrow 0$

So $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2+3}{n^3+1}$ converges.

4 pts test +/– series

$\sum +$: $\quad \sum_{n=1}^{\infty}\frac{n^2+3}{n^3+1}$

By Limit Comparison Test

Compare $a_n = \frac{n^2+3}{n^3+1}$ to $b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2+3}{n^3+1}\cdot\frac{n}{1} == \lim_{n\to\infty}\frac{n^3+3n}{n^3+1} = \frac{1}{1} = 1$

finite & positive $\Rightarrow$ behave the same

$\sum_{n=1}^{\infty}\frac{1}{n}$ diverges because Harmonic

So $\sum_{n=1}^{\infty}\frac{n^2+3}{n^3+1}$ diverges, too.

4 pts test + series

If $\sum \pm$ converges but $\sum +$ diverges, then Conditionally Convergent.

1 pt conclusion

Determine the interval of convergence of the following Power Series:

17. $$\sum_{n=1}^{+\infty} \frac{1}{(n+3)\cdot 2^n}(x-1)^n$$

[13 points]

$$|a_n| = \frac{1}{(n+3)\cdot 2^n}|x-1|^n \Rightarrow |a_{n+1}| = \frac{1}{(n+4)\cdot 2^{n+1}}|x-1|^{n+1}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^n}{2^{n+1}}\cdot\frac{n+3}{n+4}\cdot\frac{|x-1|^{n+1}}{|x-1|^n} = \lim_{n\to\infty} \frac{2^n}{2^n\cdot 2^1}\cdot\frac{n+3}{n+4}\cdot\frac{|x-1|^{n+1}}{|x-1|^n}$$

$$= \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n+3}{n+4}\cdot|x-1| = \frac{1}{2}\cdot\frac{1}{1}\cdot|x-1| = \frac{1}{2}|x-1|$$

Force limit $<1 \Rightarrow \frac{1}{2}|x-1| < 1 \Rightarrow |x-1| < 2$ 3 pts ratio test

Center : 1 Radius : 2

$\Rightarrow (1-2, 1+2) \Rightarrow (-1,3)$

Need to check endpoints $x=-1$ and $x=3$

$$@\ x=-1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(n+3)\cdot 2^n}(-1-1)^n = \sum_{n=1}^{\infty}\frac{1}{(n+3)\cdot 2^n}(-2)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+3)\cdot 2^n}\cdot 2^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n+3)}$$

By Alternatin g Series Test

1. Check decreasing

By derivative method : $a_n = \frac{1}{n+3} = (n+3)^{-1} \Rightarrow f(x) = (x+3)^{-1}$ 4 pts Alt Ser Test for +/- series

$f'(x) = -1(x+3)^{-2} = \frac{-1}{(x+3)^{23}} = \frac{-}{+} = - \Rightarrow$ Decreasing

2. Check terms $\to 0$

$\lim_{n\to\infty}\frac{1}{n+3} \Rightarrow \frac{1}{\infty} \Rightarrow 0$

So $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+3}$ converges.

$\Rightarrow$ endpoint $@\ x=-1$ can be included/c losed

$$@x=3 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(n+3)\cdot 2^n}(3-1)^n = \sum_{n=1}^{\infty}\frac{1}{(n+3)\cdot 2^n}2^n = \sum_{n=1}^{\infty}\frac{1}{k+3}$$

By Lim Comparison Test

Compare $a_n = \frac{1}{n+3} \Rightarrow b_b = \frac{1}{n}$ 4 pts Lim Comp for + series

$\lim_{n\to\infty}\frac{1}{n+3}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{n}{n+3} = \frac{1}{1} = 1 \Rightarrow$ finite & positive $\Rightarrow$ behave same

Now $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges because it' s Harmonic

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n+3}$ diverges, too.

$\Rightarrow$ endpoint $@\ x=3$ can be excluded/o pen

Interval of Convergenc e : $-1 \le x < 3 \Rightarrow [-1,3)$

2 pts final interval & endpts

18. Initially, a tank contains 200 gal of water with 10 lbs of salt dissolved in it.
A solution containing 4 lb of salt/gal is then poured into the tank at a rate of 3 gal/min and the mixed solution is drained from the tank at the same rate.

(a) Find an initial value problem whose solution is $y(t)$, the amount of salt (in lbs) at time t. (3 pts)
(b) Solve he differential equation for $y(t)$ (4pts)
(c) Determine the amount of salt in the tank after 60 min. (3 pts)

[Hint: Mixing Model: $\frac{dy}{dt} = rate\ in - rate\ out$]

rate of salt in : $\Rightarrow \frac{\text{lbs}}{\text{gal}} \cdot \frac{\text{gal}}{\text{min}} = \frac{\text{lbs}}{\text{min}} \Rightarrow 4 \cdot 3 = 12$

water level : $\Rightarrow$ no change since $[\text{rate in}] = [\text{rate out}] \Rightarrow$ always 200 gal

rate of salt out : $\Rightarrow \frac{\text{lbs}}{\text{gal}} \cdot \frac{\text{gal}}{\text{min}} = \frac{\text{lbs}}{\text{min}} \Rightarrow \frac{y}{200} \cdot 3 = \frac{3y}{200}$

$\frac{dy}{dt} = [\text{rate of salt in}] - [\text{rate of salt out}]$

$\Rightarrow \frac{dy}{dt} = [12] - \left[\frac{3y}{200}\right] \Rightarrow \frac{dy}{dt} + \frac{3}{200} y = 12$

I.F. $\Rightarrow e^{\int P} = e^{\int \frac{3}{200} dt} = e^{\frac{3}{200}t}$

$\Rightarrow e^{\frac{3}{200}t} \cdot \left(\frac{dy}{dt} + \frac{3}{200} y\right) = e^{\frac{3}{200}t} \cdot (12)$

$e^{\frac{3}{200}t} \frac{dy}{dt} + \frac{3}{200} e^{\frac{3}{200}t} y = 12 e^{\frac{3}{200}t}$

$\left(e^{\frac{3}{200}t} y\right)' = 12 e^{\frac{3}{200}t}$

$e^{\frac{3}{200}t} y = \int 12 e^{\frac{3}{200}t} dt$

$e^{\frac{3}{200}t} y = 12 \cdot \frac{200}{3} e^{\frac{3}{200}t} + C = 800 e^{\frac{3}{200}t} + C$

$y = 800 + C e^{-\frac{3}{200}t}$

3 pts setup eq & 4 pts DE solution

$y = 800 + C e^{-\frac{3}{200}t}$

$@ (0,10) \Rightarrow 10 = 800 + C e^{0} \Rightarrow C = -790$

$y = 800 - 790 e^{-\frac{3}{200}t}$

$@\ t = 60 \Rightarrow y = 800 - 790 e^{-\frac{3}{200} 60}$

$y = 800 - 790 e^{-\frac{9}{10}} \approx 478.81$

3 pts eval (don't have to turn to decimal – exponent is okay)

Direct Comparison Test:

If $a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If $b_n \le a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Limit Comparison Test: If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ and $L > 0$ and finite, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ behave the same. That is, both series converge or both diverge.

Divergence Test: In order for a series $\sum_{n=0}^{\infty} a_n$ to converge, the terms a_n must be going to zero. Just because the terms in the series are going to zero does not automatically mean that it converges. However, if they go to any other limit besides zero, the series definitely *diverges*:

If $\sum_{n=0}^{\infty} a_n$ and $\lim_{n\to\infty} a_n = L \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Integral Test: Let $\sum_{n=1}^{+\infty} u_n$ be a series with positive terms, and let *f(x)* be the function that results when *n* is replaced by *x*. If f is decreasing and continuous on the interval $[1, +\infty)$, then either $\sum_{n=1}^{+\infty} u_n$ and $\int_1^{+\infty} f(x)dx$ both converge or both diverge.

Alternating Series Test for either form $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ or $\sum_{n=1}^{\infty} (-1)^n a_n$

Check that the following two conditions are met:

(1) Ignoring the $+/-$ signs, check that the terms are decreasing. That is, make certain that $a_n \ge a_{n+1}$

(2) Check that the terms are heading to zero. That is, make sure that $\lim_{n\to\infty} a_n = 0$

If *both* conditions are satisfied, then the alternating series converges.

Ratio Test for Absolute Convergence:

If $\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = L$ and $L < 1$, then the series $\sum_{n=1}^{\infty} u_n$ converges absolutely. If $L > 1$, then the series diverges.

If $L = 1$, then the test provides no useful information and another test must be used.

Root Test for Absolute Convergence:

If $\lim_{n\to\infty} \sqrt[n]{|u_n|} = L$ and $L < 1$, then the series $\sum_{n=1}^{\infty} u_n$ converges absolutely. If $L > 1$, then the series diverges.

If $L = 1$, then the test provides no useful information and another test must be used.

A series $\sum_{n=1}^{\infty} u_n$ *converges absolutely* if $\sum_{n=1}^{\infty} |u_n|$ converges.

A series $\sum_{n=1}^{\infty} u_n$ *converges conditionally* if $\sum_{n=1}^{\infty} |u_n|$ diverges and $\sum_{n=1}^{\infty} u_n$ converges.