Name ____ MTH 1001 - Calculus 1

PRACTICE ADVANCED STANDING EXAM

1. (a) Write the general *definition of the derivative* for a function $f(x)$

[14pts]

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ [4 pt setup]

(b) Find $f'(x)$ by using the *definition of the derivative* with the following function:

$f(x) = \frac{1}{x}$

$f'(x) = \lim_{h\to 0} \frac{\left[\frac{1}{x+h}\right]-\left[\frac{1}{x}\right]}{h}$ [3 pt setup]

$= \lim_{h\to 0} \frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h} = \lim_{h\to 0} \frac{\frac{x-x-h}{x(x+h)}}{h}$ [3 pt expand & cancel]

$= \lim_{h\to 0} \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}} = \lim_{h\to 0} \left(\frac{-h}{x(x+h)} \cdot \frac{1}{h}\right)$ [2 pts factor & simplify]

$= \lim_{h\to 0} \left(\frac{-1}{x(x+h)}\right) = \frac{-1}{x(x+0)} = \frac{-1}{x^2}$ [2 pts take lim & answer]

2. Find the derivative: $f(x) = x^3 \tan(2x-1)$

[12pts]

$f'(x) = 3x^2 \cdot \tan(2x-1) + x^3 \cdot \sec^2(2x-1) \cdot 2$

[3 pts prod rule & 3 pts poly deriv & 3 pts trig derive & 3 pts chain rule]

3. Find the derivative: $f(x) = e^{x^3} + \ln(\sec x) + \csc(\ln x)$

[9pts]

3 pts *e*-term & 3 pts log term & 3 pts last trig term

$f'(x) = e^{x^3} \cdot 3x^2 + \frac{\sec x \tan x}{\sec x} - \csc(\ln x)\cot(\ln x) \cdot \frac{1}{x} = 3x^2 e^{x^3} + \tan x - \frac{\csc(\ln x)\cot(\ln x)}{x}$

(don't need to simplify any terms)

4. Find the derivative $\frac{dy}{dx}$ for the following: $x^2 + y^3 = ye^{5x}$

[10pts]

$$2x + 3y^2 \frac{dy}{dx} = \frac{dy}{dx} \cdot e^{5x} + y \cdot 5e^{5x}$$
$$3y^2 \frac{dy}{dx} - e^{5x} \frac{dy}{dx} = 5ye^{5x} - 2x$$
$$\left(3y^2 - e^{5x}\right)\frac{dy}{dx} = 5ye^{5x} - 2x$$
$$\frac{dy}{dx} = \frac{5ye^{5x} - 2x}{3y^2 - e^{5x}}$$

[6 pts derive & 2 pts rearrange & 2 pts ans]

5. Find the derivative: $f(x) = \frac{\sin x}{x} + \sin^{-1} x + \sinh x$

[9pts]

3 pts quot rule & 3 pts inv trig & 3 pts hyperbolic trig

$$f'(x) = \frac{x\cos x - \sin x}{x^2} + \frac{1}{\sqrt{1-x^2}} + \cosh x$$

6. Find the derivative: $y = x^{3x}$

[12 pts]

$$y = x^{3x}$$
$$\ln y = \ln\left(x^{3x}\right)$$ 2 pts
$$\ln y = 3x \ln(x)$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = 3\ln(x) + 3x \cdot \frac{1}{x}$$ 2 pts
$$\frac{dy}{dx} = y(3\ln(x) + 3)$$
$$\frac{dy}{dx} = x^{3x}(3\ln(x) + 3)$$ 2 pts

7. A toy car moves along a straight track during time $0 \le t \le 4$. It's position at any
[10pts] time from a fixed point along the track is given by $s(t) = t^3 - 3t^2$
Answer the following about the motion of the car.
(Note: The time *t* is measured in minutes and distance *s* in inches.)

(a) What is the position, velocity, and acceleration of the car at the time *t* = 3 minutes?

$s(t) = t^3 - 3t^2 \underset{\text{at } t=3}{\Rightarrow} 27 - 27 = 0$ [2pts]

$v(t) = 3t^2 - 6t \underset{\text{at } t=3}{\Rightarrow} 27 - 18 = +9$ [2pts]

$a(t) = 6t - 6 \underset{\text{at } t=3}{\Rightarrow} 18 - 6 = +12$ [2pts]

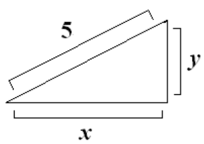
(b) At what time does the car come to a stop?

$v(t) = 0 \Rightarrow 3t^2 - 6t = 3t(t-2) = 0 \text{ at } t = 0 \text{ and } t = 2$ [4pts]

8. A 5 ft ladder is leaning against a wall and starts to slide. How fast is the bottom
[12pts] edge of the ladder moving along the floor when the top corner of the ladder is 3 ft up the wall and sliding down the wall at a rate of 8 ft/sec?

$x^2 + y^2 = 5^2$ [2 pts]

$x^2 + y^2 = 25 \Rightarrow x^2 + 9 = 25 \Rightarrow x^2 = 16 \Rightarrow x = 4$



$x = 4 \quad \frac{dx}{dt} = ?$

$y = 3 \quad \frac{dy}{dt} = -8$

$x^2 + y^2 = 25$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ [5 pts]

$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$[4]\frac{dx}{dt} + [3][-8] = 0$

$4\frac{dx}{dt} = 24$ [5 pts]

$\frac{dx}{dt} = +6\,\frac{\text{ft}}{\text{sec}}$

9. Use L'Hôpital's Rule to evaluate the following limit:
[8pts]

$\lim_{x\to 0} \frac{x^3 + 5\sin x}{x\cos x} \underset{\frac{0}{0}}{\Rightarrow} \lim_{x\to 0} \frac{3x^2 + 5\cos x}{1\cos x - x\sin x} = \frac{0+5}{1-0} = \frac{5}{1} = 5$ 6 pts L'Hôp [3 pts num & 3 pts denom] & 2 pts ans

10. Graph the following Rational Function:

$$f(x)=\frac{36(x-1)}{x^2}$$

[16 pts]

Hint: $f'(x)=\frac{36(2-x)}{x^3}$ and $f''(x)=\frac{72(x-3)}{x^4}$

(Use calculus to find the locations of any important points [maxs, mins, pts of inflection] and label them on the graph.)

$$f'(x)=\frac{36(2-x)}{x^3}=\frac{AB}{C}$$

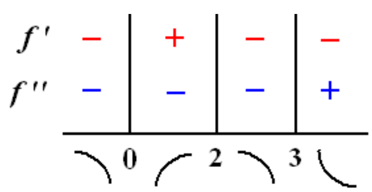
A	+		+		+
B	+		+		−
C	−		+		+
	−	0	+	2	−

[2 pt first deriv & chart]

$$f''(x)=\frac{72(x-3)}{x^4}=\frac{AB}{C}$$

A	+		+		+
B	−		−		+
C	+		+		+
	−	0	−	3	+

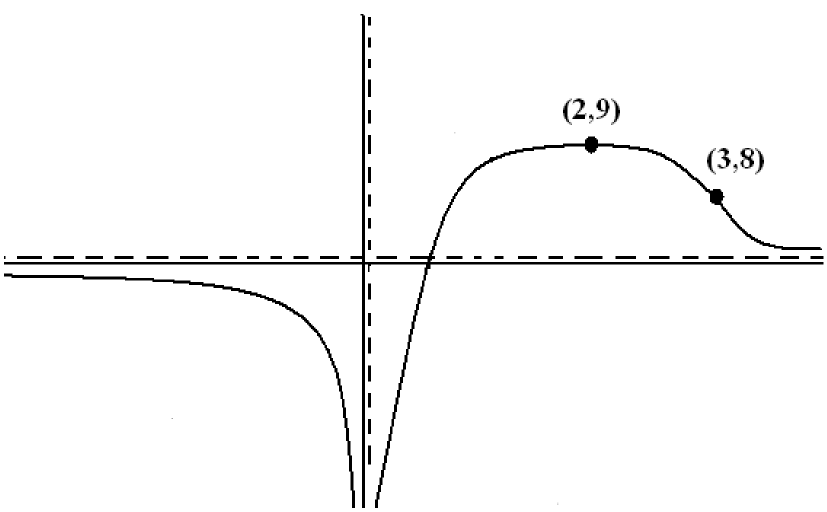
[2 pt second deriv & chart]



[3 pts assemble pieces]

$$@x=2\Rightarrow\frac{36(2-1)}{2^2}=\frac{36\cdot1}{4}=9\Rightarrow(2,9)$$
$$@x=3\Rightarrow\frac{36(3-1)}{3^2}=\frac{36\cdot2}{9}=4\cdot2=8\Rightarrow(3,8)$$

[1 pt get pts]

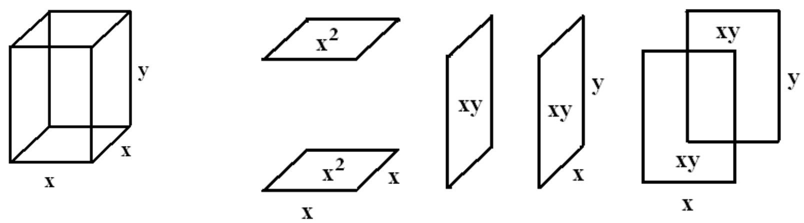


[4 pts H.A. & V.A.]

[4 pts final picture]

11. A box with a closed top is going to be manufactured so that its base is a square and its volume will be $100\,\text{cm}^3$. If the material to make the top and bottom of the box cost $50 per square cm and the material for the sides costs $4 per square cm, find the dimensions that will minimize the cost of the box.

[12 pts]



$x^2 y = 100$ [2 pts constraint eq]

$y = \frac{100}{x^2}$ [1 pt rewrite]

$C = 50(2x^2) + 4(4xy) = 100x^2 + 16xy$ [2 pts cost]

$C = 100x^2 + 16x\left(\frac{100}{x^2}\right) = 100x^2 + 1600x^{-1}$ [2 pts rewrite]

$C = 200x - 1600x^{-2} = \frac{200x^3 - 1600}{x^2}$ [2 pts derive]

$200x^3 - 1600 = 0 \Rightarrow x^3 = \frac{1600}{200} = \frac{16}{2} = 8 \Rightarrow x = \sqrt[3]{8} = 2$

C.P. : $x = 2$ and $x = 0 \Rightarrow$ discard $x = 0$

$x = 2$ [2 pts x-value]

$y = \frac{100}{4} = 25$ [1 pt y-value]

12. Find the exact area under the curve $f(x) = 2x + 1$ over the interval $[a, b]$, where x_i is the right endpoint of each equal subinterval, given $a = 1$ and $b = 3$.

[16pts]

Hint – Evaluate the limit: $\lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$\sum_{i=1}^{n}(1) = n$ $\sum_{i=1}^{n}(i) = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n}(i^2) = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{n}(i^3) = \left[\frac{n(n+1)}{2}\right]^2$

$\Delta x = \frac{3-1}{n} = \frac{2}{n}$ [2 pts]

$x_i = 1 + i \cdot \frac{2}{n} = 1 + \frac{2}{n} i$

$y = f(x_i) = 2\left(1 + \frac{2}{n} i\right) + 1 = 2 + \frac{4}{n} i + 1 = 3 + \frac{4}{n} i$ [4 pts]

$$Area = \lim_{n \to \infty}\left(\sum_{i=1}^{n} f(x_i)\Delta x\right) = \lim_{n \to \infty}\left(\sum_{i=1}^{n}\left[\left(3 + \frac{4}{n} i\right)\frac{2}{n}\right]\right) = \lim_{n \to \infty}\left(\sum_{i=1}^{n}\left[\frac{6}{n} + \frac{8}{n^2} i\right]\right)$$

$$= \lim_{n \to \infty}\left(\sum_{i=1}^{n}\left[\frac{6}{n}\right] + \sum_{i=1}^{n}\left[\frac{8}{n^2} i\right]\right) = \lim_{n \to \infty}\left(\frac{6}{n} \cdot \sum_{i=1}^{n}[1] + \frac{8}{n^2} \cdot \sum_{i=1}^{n}[i]\right) = \lim_{n \to \infty}\left(\frac{6}{n} \cdot n + \frac{8}{n^2} \cdot \frac{n(n+1)}{2}\right)$$

$$= \lim_{n \to \infty}\left(6 + \frac{4(n+1)}{n}\right) = \lim_{n \to \infty}\left(6 + \frac{4n+4}{n}\right) = 6 + \frac{4}{1} = 6 + 4 = 10$$

[2 pts setup lim & 4 pts breakup sigmas & 4 pts final ans]

13. Evaluate the indefinite integral: $\int \left(50x^4 + 10x^3 + 12\sqrt{x}\right)dx$

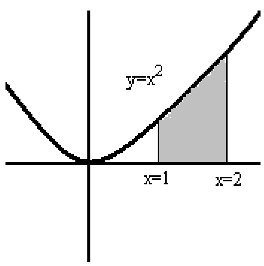
[12pts]

$$\int \left(50x^4 + 10x^3 + 12x^{\frac{1}{2}}\right)dx = 10x^5 + \tfrac{5}{2}x^4 + 8x^{\frac{3}{2}} + C$$

[1 pt rewrite root & 2 pts x^5-term & 4 pts x^4-term & 4 pts $x^{3/2}$-term & 1 pt "C"]

Setup a definite integral and find the area of the indicated regions:

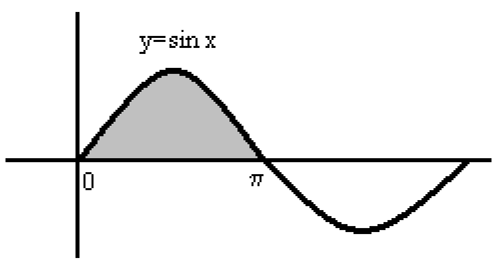
14.
[5pts]



$$\int_1^2 x^2 dx = \left(\tfrac{1}{3}x^3\right)\Big]_1^2 = \frac{(2)^3}{3} - \frac{(1)^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

2 pts setup & 2 pts antiderive & 1 pt plugin/answer

15.
[5pts]



$$\int_0^{\pi} \sin x dx = (-\cos x)\Big]_0^{\pi} = (-\cos(\pi)) - (-\cos(0)) = (-[-1]) - (-1) = 1 + 1 = 2$$

1 pt setup & 2 pts antiderive & 2 pts plugin/answer

16. Evaluate the following: $\frac{d}{dx}\left(\int_4^{x^3} e^{T^2} dT\right)$

[10pts] $e^{(x^3)^2} \cdot 3x^2 - e^{(4)^2} \cdot 0 = e^{x^6} \cdot 3x^2 - 0 = 3x^2 e^{x^6}$

6 pts cancel and try to plugin lim & 4 pts remember to plugin to *dt*

17. Evaluate the indefinite integral: $\int \left[24\sin^2(4x)\cos(4x)\right]dx$

[14pts] 2 pts *u*-choice

$u = \sin(4x)$ $du = 4\cos(4x)dx$ $dx = \frac{du}{4\cos(4x)}$

6 pts rewrite/simplify 4 pts *u*-antideriv 2 pts back to *x*'s

$$\int 24(\sin(4x))^2 \cos(4x)dx = \int 24u^2 \cos(4x)\tfrac{du}{4\cos(4x)} = \int \tfrac{24}{4}u^2 du = \int 6u^2 du = 2u^3 + C = 2(\sin(4x))^3 + C$$

18. Evaluate the definite integral: $\int_0^1 \left[8x(x^2+1)^3\right]dx$

[14pts]

1 pt *u*-choice

$u = x^2 + 1$ $du = 2xdx$ $dx = \frac{du}{2x}$ $x = 1 \Rightarrow u = 1^2 + 1 = 2$, $x = 0 \Rightarrow u = 0^2 + 1 = 1$

5 pts rewrite/simplify 3 pts *u*-antideriv 3 pts change lim or back to *x*'s 2 pts ans

$$\int_{x=0}^{x=1} \left[8x(x^2+1)^3\right]dx = \int_{u=1}^{u=2} \left[8xu^3\right]\tfrac{du}{2x} = \int_{u=1}^{u=2} \left[4u^3\right]du = u^4\Big]_{u=1}^{u=2} = (2^4) - (1^4) = 16 - 1 = 15$$